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PSEUDO-HERMITIAN SYMMETRIES

BY

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ABSTRACT

In this paper, we study different types of symmetries for the Tanaka–Webster connection of contact strictly pseudo-convex pseudo-Hermitian CR manifolds.

1. Introduction

A **contact manifold** (M,η) is a smooth manifold M^{2n+1} together with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. This means that $d\eta$ has maximal rank $2n$ on the contact distribution $D = \ker \eta$. Given a contact manifold, we can consider two associated structures. One is an associated Riemannian metric g and we obtain a **contact Riemannian manifold** $(M;\eta,g)$. The other is a **pseudo-Hermitian** and **strictly pseudo-convex structure** (η, L), where L is the **Levi form** associated with an endomorphism J on D such that $J^2 = -I$. Here, J defines an almost CR structure $\mathcal{H} = \{X - iJX : X \in D\}$. We

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obtain a **contact strictly pseudo-convex, pseudo-Hermitian manifold** (or **almost CR manifold**) $(M;\eta,L)$. There is a one-to-one correspondence between the two associated structures by the relation

$$g = L + \eta\otimes\eta,$$

where we denote by the same letter L the natural extension ($i_\xi L = 0$) of the Levi form to a (0,2)-tensor field on M. So, contact Riemannian structures go hand in hand with contact strictly pseudo-convex almost CR structures. For theoretical considerations, it is desirable to have integrability of the almost complex structure J on D. If this is the case, we speak of an (**integrable**) **CR structure** and of a **CR manifold.** When we look at a contact manifold from the Riemannian point of view, i.e., we consider the contact metric manifold $(M;\eta,g)$, the presence of (metric) symmetries is an important topic. In particular, the question comes up when a contact metric manifold is locally symmetric, i.e., when its Riemannian curvature tensor R satisfies

$$\nabla R = 0.$$

Quite recently, the authors have proved in [13] that a locally symmetric contact metric space is either Sasakian and of constant curvature 1 or locally isometric to the unit tangent sphere bundle of a Euclidean space with its standard contact metric structure. (For Sasakian manifolds, the situation had been clear already for a long time by work of Okumura [24].) This result means that local symmetry is too strong a condition to impose in contact geometry. For this reason, T. Takahashi ([27]) introduced **Sasakian locally φ-symmetric spaces,** which may be considered as the analogues of locally Hermitian symmetric spaces. He calls a Sasakian manifold locally φ-symmetric if the Riemannian curvature tensor R satisfies

$$(*) \qquad g((\nabla_X R)(Y,Z)V,U) = 0$$

for all vector fields X,Y,Z,V and U orthogonal to ξ. He proves that this condition is equivalent to having φ-geodesic symmetries which are local automorphisms. Later, it was proved in [8] that the isometry property of the φ-geodesic symmetry is already sufficient. For the broader class of contact Riemannian manifolds, we have two generalizations for the notion of local φ-symmetry. In [7], a contact Riemannian manifold is called locally φ-symmetric if it satisfies the same curvature condition $(*)$ as in the Sasakian case. This is a very workable definition from the technical point of view, but it is still unclear what it means geometrically when the contact manifold is not Sasakian or K-contact. In [14], the authors give a different definition for a locally φ-symmetric contact Riemannian manifold: they require the characteristic reflections (i.e., the reflections with respect to the integral curves of ξ) to be local isometries. This geometric definition leads to an infinite number of curvature conditions, including $(*)$. The first type of symmetry is therefore called **local φ-symmetry in the weak sense** and the second type **local φ-symmetry in the strong sense.**

When we look at a contact manifold from the point of view of its (almost) CR structure, there exists a canonical affine connection, different from the Levi Civita connection of an associated metric, and invariant under D-homothetic deformations. This is the **Tanaka–Webster connection** $\hat{\nabla}$ on a strictly pseudo-convex CR manifold. In earlier work [16], [17], [18], the second author started the intriguing study of the interactions between the contact Riemannian structure and the contact strictly pseudo-convex (almost) CR structure, with the Tanaka–Webster connection playing a major role. In this context, we now want to define and study analogues of the different types of symmetry mentioned above for contact metric spaces. As the pseudo-Hermitian counterpart of local symmetry, we introduce **Tanaka–Webster parallel spaces**: these are strictly pseudo-convex CR manifolds whose Tanaka–Webster torsion tensor $\hat{T}$ and Tanaka–Webster curvature tensor $\hat{R}$ are parallel with respect to $\hat{\nabla}$:

$$\hat{\nabla}\hat{T} = 0, \quad \hat{\nabla}\hat{R} = 0.$$

We classify such spaces completely in Section 4. Next, in an analogous way as locally φ-symmetric contact Riemannian spaces in the strong sense, we define **strongly locally pseudo-Hermitian symmetric spaces** by the following property: all characteristic $\hat{\nabla}$-reflections are affine mappings, i.e., they preserve the Tanaka–Webster connection $\hat{\nabla}$. We found by chance that it was already introduced and investigated (cf. [3] or [19]) under another name, a locally sub-symmetric space, among the studies of the sub-Riemannian geometry. But, along our own context we determine strongly locally pseudo-Hermitian symmetric spaces in the last part of Section 4.

At the final stage, we define **locally pseudo-Hermitian symmetric spaces in the weak sense** to be strictly pseudo-convex CR manifolds whose Tanaka–Webster curvature tensor $\hat{R}$ satisfies

$$g((\hat{\nabla}_X \hat{R})(Y,Z)V,U) = 0$$

for all vector fields X, Y, Z, V and U orthogonal to ξ. (They correspond to local φ-symmetry in the weak sense.) In Section 5, we give examples of such manifolds and study the three-dimensional ones in some detail. We also show that this class of spaces is essentially different from the weakly locally φ-symmetric contact metric manifolds.

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. We start by collecting some fundamental material about contact Riemannian geometry and contact strictly pseudo-convex CR manifolds. We refer to [5] for further details.

A $(2n+1)$-dimensional manifold M^{2n+1} is a **contact manifold** if it is equipped with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, the **characteristic vector field,** satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there also exists a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\begin{gathered} g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad d\eta(X,Y) = g(X, \varphi Y), \\ \varphi^2 X = -X + \eta(X)\xi, \end{gathered} \tag{1}$$

where X and Y are vector fields on M. From (2), it follows that

$$\varphi\xi = 0, \ \eta \circ \varphi = 0, \ \eta(X) = g(X, \xi). \tag{2}$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2) is said to be a **contact Riemannian manifold** or **contact metric manifold** and is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define a $(1,1)$-tensor field h by $h = \frac{1}{2} L_\xi \varphi$, where L denotes Lie differentiation. The operator h is symmetric and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h, \tag{3}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{4}$$

$$\begin{aligned} g(R(X,Y)\xi, Z) =& g((\nabla_Y \varphi)X - (\nabla_X \varphi)Y, Z) + g((\nabla_Y \varphi h)X \\ &- (\nabla_X \varphi h)Y, Z), \end{aligned} \tag{5}$$

for all vector fields X, Y, Z on M, where ∇ is the Levi-Civita connection and R the Riemannian curvature tensor.

A contact Riemannian manifold for which ξ is a Killing vector field, is called a **K-contact manifold.** It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\varphi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be **normal** or **Sasakian.** We note that every Sasakian manifold is also K-contact, but the converse is only true in dimension 3. There are several equivalent conditions for integrability of J in terms of the structure tensors on M. The following two will be needed further on:

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{6}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{7}$$

for all vector fields X and Y on M.

Next, we recall the natural relation of contact metric manifolds with CR manifolds. For a contact Riemannian manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as the direct sum $T_pM = D_p \oplus \{\xi\}_p$, where we denote $D_p = \{v \in T_pM \colon \eta(v) = 0\}$. Then $D : p \to D_p$ defines a $2n$-dimensional distribution orthogonal to ξ, which is called the **contact distribution** or the **contact subbundle**. For a given contact Riemannian manifold $M = (M; \eta, g)$, its associated almost CR-structure is given by the holomorphic subbundle

$$\mathcal{H} = \{X - iJX : X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM, where $J = \varphi|D$, the restriction of φ to D. We see that each fiber $\mathcal{H}_x$, $x \in M$, is of complex dimension n, $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$ and $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$.

We define the **Levi form** L by

$$L : D \times D \to \mathcal{F}(M), \quad L(X,Y) = -d\eta(X, JY)$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. Since $d\eta(X,Y) = g(X,\varphi Y)$ on a contact metric manifold, the Levi form is Hermitian and positive definite. So, the pair (η, L) is a **strictly pseudo-convex (pseudo-Hermitian) almost CR structure** on M.

The associated CR structure is **integrable** if $[\mathcal{H},\mathcal{H}] \subset \mathcal{H}$. This property does not hold for a general contact metric manifold. In terms of the structure tensors, integrability is equivalent to the condition $\Omega = 0$, where Ω is the $(1,2)$-tensor field on M defined as

$$\Omega(X,Y) = (\nabla_X\varphi)Y - g(X+hX,Y)\xi + \eta(Y)(X+hX) \tag{8}$$

for vector fields X, Y on M (see Proposition 4 in Section 3). In this case, the pair (η, L) is called a **strictly pseudo-convex (integrable) CR structure** and $(M;\eta,L)$ is called a **strictly pseudo-convex CR manifold.** From (6) and (8), we see that the associated CR structure of a Sasakian manifold is strictly pseudo-convex integrable (cf. [20]). The same is true for the associated CR structure of any three-dimensional contact metric space.

A **pseudo-homothetic** or **D-homothetic transformation** of a contact metric manifold [29] is a change of structure tensors of the form

$$\bar\eta = a\eta,\ \ \bar\xi = 1/a\ \xi,\ \ \bar\varphi = \varphi, \quad \bar g = ag + a(a-1)\eta\otimes\eta, \tag{9}$$

where a is a positive constant. From (9), we have $\bar h = (1/a)h$. By using the well-known Koszul formula

$$\begin{aligned}2g(\nabla_X Y, Z) =& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)\\ &- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),\end{aligned}$$

we have

$$\bar\nabla_X Y = \nabla_X Y + C(X,Y), \tag{10}$$

where C is the (1,2)-type tensor defined by

$$C(X,Y) = -(a-1)[\eta(Y)\varphi X + \eta(X)\varphi Y] - \frac{a-1}{a}g(\varphi hX, Y)\xi.$$

Remark 1: Integrability of the associated CR structure is preserved under pseudo-homothetic transformations. In fact, by direct computations, we have

$$(\bar\nabla_X\bar\varphi)Y = (\nabla_X\varphi)Y + (a-1)\eta(Y)\varphi^2X - (a-1)/ag(X,hY)\xi.$$

From this, we easily see that $\Omega = 0$ implies $\bar\Omega = 0$.

In what follows, an important role will be played by a specific class of contact metric manifolds, namely those for which it holds

$$R(X,Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y), \tag{11}$$

where I denotes the identity transformation and $k, \mu \in \mathbb{R}$. These spaces are called (k, μ)**-spaces** and were introduced in [6]. As examples, we have Sasakian spaces ($k = 1$ and $h = 0$) and also the unit tangent sphere bundles of spaces of constant curvature c ($k = c(2-c)$ and $\mu = -2c$). Since the unit tangent sphere bundle is non-Sasakian when $c \neq 1$ [31], this gives us a lot of non-Sasakian examples.

It is a surprising fact that the condition (11) completely determines the curvature tensor in the non-Sasakian case. Indeed, the following theorem holds.

THEOREM 1: *Let $(M;\eta,g)$ be a non-Sasakian (k,μ)-space. Then $k < 1$ and the curvature tensor R is given explicitly by*

$$\begin{aligned} R(X,Y)Z =& (1-\mu/2)(g(Y,Z)X - g(X,Z)Y) \\ &+ g(Y,Z)hX - g(X,Z)hY - g(hX,Z)Y + g(hY,Z)X \\ &+ \frac{1-(\mu/2)}{1-k}(g(hY,Z)hX - g(hX,Z)hY) \\ &- \frac{\mu}{2}(g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y) \\ &+ \frac{k-(\mu/2)}{1-k}(g(\varphi hY,Z)\varphi hX - g(\varphi hX,Z)\varphi hY) + \mu g(\varphi X,Y)\varphi Z \\ &+ \eta(X)((k-1+(\mu/2))g(Y,Z) + (\mu-1)g(hY,Z))\xi \\ &- \eta(Y)((k-1+(\mu/2))g(X,Z) + (\mu-1)g(hX,Z))\xi \\ &- \eta(X)\eta(Z)((k-1+(\mu/2))Y + (\mu-1)hY) \\ &+ \eta(Y)\eta(Z)((k-1+(\mu/2))X + (\mu-1)hX) \end{aligned} \tag{12}$$

for all vector fields X, Y, Z on M.

Actually, the complete local geometry is determined by the condition (11), as is shown in [10], where a full local classification of (k,μ)-spaces is presented.

We finish the introduction by recalling some of the properties of (k,μ)-spaces which we will make use of further on. Firstly, as proved in [6], the class of (k,μ)-spaces is invariant under pseudo-homothetic transformations. More precisely, a pseudo-homothetic transformation with constant a changes (k,μ) into $(\bar{k},\bar{\mu})$, where

$$\bar{k} = \frac{k + a^2 - 1}{a^2} \ , \quad \bar{\mu} = \frac{\mu + 2a - 2}{a} \ . \tag{13}$$

Remark 2: From these formulas, we see that the values $k = 1$ and $\mu = 2$ are preserved under D-homothetic transformations. The case $k = 1$ corresponds to the class of Sasakian manifolds; for the case $\mu = 2$, we will find a geometric interpretation further on (see Propositions 10 and 11, and Theorem 12).

Secondly, the associated CR structure of a (k, μ)-space is integrable, i.e., these spaces are contact strictly pseudo-convex CR manifolds. This gives us an expression for $\nabla\varphi$ via (8). Moreover, also for ∇h, we have an explicit formula [6]

$$\begin{aligned}(\nabla_X h)Y =& [(1 - k)g(X, \varphi Y) - g(X, \varphi hY)]\xi \\ & - \eta(Y)[(1 - k)\varphi X + \varphi hX] - \mu\eta(X)\varphi hY.\end{aligned} \tag{14}$$

Note that this allows to write down explicit expression also for ∇R and for higher order covariant derivatives of all structure tensors. This was instrumental in obtaining the local classification mentioned earlier. From (14), it follows immediately that a (k, μ)-space satisfies $g((\nabla_X h)Y, Z) = 0$ for all vector fields X, Y, Z orthogonal to ξ, i.e., it is an **η-parallel contact metric space.** Quite recently, the present authors proved that also the converse holds:

THEOREM 2 ([12]): *An η-parallel contact metric space is a K-contact space or a (k, μ)-space.*

Finally, (k, μ)-spaces have nice geometrical properties.

THEOREM 3 ([9]): *A non-Sasakian (k, μ)-space is (locally) contact-homogeneous and locally φ-symmetric (in the strong sense and hence also in the weak sense).*

3. The Tanaka–Webster connection

Now, we review the **generalized Tanaka–Webster connection** $\hat{\nabla}$ on a contact strictly pseudo-convex almost CR manifold $M = (M; \eta, L)$ ([30]). It is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (4), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + A(X,Y), \tag{15}$$

where we put

$$A(X,Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi h X) - g(\varphi X + \varphi h X, Y)\xi. \tag{16}$$

We see that the generalized Tanaka–Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(X,Y) = 2g(X,\varphi Y)\xi + \eta(Y)\varphi h X - \eta(X)\varphi h Y. \tag{17}$$

In particular, for a K-contact Riemannian manifold we get

$$A(X,Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{18}$$

The generalized Tanaka–Webster connection can also be characterized differently.

Proposition 4 ([30]): *The generalized Tanaka–Webster connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M;\eta,g)$ is the unique linear connection satisfying the following conditions:*

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$;
(iii-1) $\hat{T}(X,Y) = 2L(X,JY)\xi$, $X,\ Y \in D$;
(iii-2) $\hat{T}(\xi,\varphi Y) = -\varphi\hat{T}(\xi,Y)$, $Y \in D$;
(iv) $(\hat{\nabla}_X\varphi)Y = \Omega(X,Y)$, $X,\ Y \in TM$.

We note that the Tanaka–Webster connection ([28], [33]) was originally defined for a nondegenerate integrable CR manifold, in which case condition (iv) reduces to $\hat{\nabla}J = 0$. The above definition is a natural generalization to the non-integrable case (see also [2]).

Proposition 5: *The (generalized) Tanaka–Webster connection is pseudo-homothetically invariant.*

Proof. From (8), (9) and (10) we have

$$\bar{\hat{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{A}(X,Y) = \nabla_X Y + C(X,Y) + \bar{A}(X,Y)$$

and

$$\bar{A}(X,Y) = a(\eta(X)\varphi Y + \eta(Y)\varphi X) + \eta(Y)\varphi h X - g(\varphi X, Y)\xi - \frac{1}{a}g(\varphi h X, Y)\xi.$$

Together with the definition of the tensor C, we see that $C(X,Y)+\bar{A}(X,Y)=A(X,Y)$. Hence, it follows that the generalized Tanaka–Webster connection is pseudo-homothetically invariant. ∎

Corollary 6: *The Tanaka–Webster curvature tensor $\hat{R}$, its torsion tensor $\hat{T}$ and their covariant derivatives $\hat{\nabla}\hat{R}$ and $\hat{\nabla}\hat{T}$ are pseudo-homothetically invariant.*

We look at $\hat{R}(X,Y)Z=\hat{\nabla}_X(\hat{\nabla}_Y Z)-\hat{\nabla}_Y(\hat{\nabla}_X Z)-\hat{\nabla}_{[X,Y]}Z$ in some more detail for the case when the CR structure is integrable, i.e., $\hat{\nabla}\varphi=0$. First we have quite generally

Proposition 7:

$$\hat{R}(X,Y)Z=-\hat{R}(Y,X)Z,\quad L(\hat{R}(X,Y)Z,W)=-L(\hat{R}(X,Y)W,Z).$$

The first identity follows trivially from the definition of $\hat{R}$. Since the connection is metric with respect to its associated metric g, $\hat{\nabla}g=0$, the second identity is proved in a similar way as for the case of Riemanian curvature tensor. Since the Tanaka–Webster connection is not torsion-free, the Jacobi- or Bianchi-identies do not hold, in general. Before we study the curvature tensor $\hat{R}$, from (3), (15) and (16) we have

$$\begin{aligned}(\hat{\nabla}_X h)Y=&(\nabla_X h)Y+A(X,hY)-hA(X,Y)\\ =&(\nabla_X h)Y+2\eta(X)\varphi hY+g((\varphi h+\varphi h^2)X,Y)\xi\\ &+\eta(Y)(\varphi hX+\varphi h^2X).\end{aligned}\tag{19}$$

From the definition of $\hat{R}$, together with (15), taking account of $\hat{\nabla}\eta=0$, $\hat{\nabla}\xi=0$, $\hat{\nabla}g=0$, $\hat{\nabla}\varphi=0$ and (19), straightforward computations yield

$$\begin{aligned}&\hat{R}(X,Y)Z\\ =&R(X,Y)Z+\eta(Z)\big(\varphi P(X,Y)+\varphi(A(X,hY)-A(Y,hX))\\ &\qquad-\varphi h(A(X,Y)-A(Y,X))\big)\\ &-g\big(\varphi P(X,Y)+\varphi(A(X,hY)-A(Y,hX))-\varphi h(A(X,Y)-A(Y,X)),Z\big)\xi\\ &-2g(\varphi X,Y)A(\xi,Z)-\eta(X)A(\varphi hY,Z)+\eta(Y)A(\varphi hX,Z)\\ &-\eta(X)\varphi A(Y,Z)+\eta(Y)\varphi A(X,Z)\\ &+\eta(A(X,Z))(\varphi Y+\varphi hY)-\eta(A(Y,Z))(\varphi X+\varphi hX)\\ &+g(\varphi X+\varphi hX,A(Y,Z))\xi-g(\varphi Y+\varphi hY,A(X,Z))\xi\end{aligned}$$

where we put $P(X,Y) = (\nabla_X h)Y - (\nabla_Y h)X$. By using (2), (2), (3) and (16), we have

$$\hat{R}(X,Y)Z = R(X,Y)Z + B(X,Y)Z, \tag{20}$$

where

$$\begin{aligned} B(X,Y)Z =& \eta(Z)\varphi P(X,Y) - g(\varphi P(X,Y),Z)\xi \\ & - \eta(Z)[\eta(Y)(X+hX) - \eta(X)(Y+hY)] \\ & + \eta(Y)g(X+hX,Z)\xi - \eta(X)g(Y+hY,Z)\xi \\ & + g(\varphi Y+\varphi hY,Z)(\varphi X+\varphi hX) - g(\varphi X+\varphi hX,Z)(\varphi Y+\varphi hY) \\ & - 2g(\varphi X,Y)\varphi Z \end{aligned}$$

for all vector fields X, Y, Z in M.

4. Tanaka–Webster parallel spaces

As an analogue of locally symmetric contact metric spaces, we now introduce Tanaka–Webster parallel spaces.

Definition 1: A contact metric space is a **Tanaka–Webster parallel space** (T.-W. parallel space, for short) if its Tanaka–Webster torsion tensor $\hat{T}$ and its curvature tensor $\hat{R}$ satisfy

$$\hat{\nabla}\hat{T} = 0, \quad \hat{\nabla}\hat{R} = 0.$$

In [22], S. Kobayashi and K. Nomizu call a connection **invariant by parallelism** if for any pair of points p and q in M and for any curve γ from p to q, there exists a (unique) local affine isomorphism f such that $f(p) = q$ and such that the differential of f at p coincides with the parallel displacement $\tau_\gamma : T_pM \to T_qM$ along γ. By [22, Corollary 7.6], this is equivalent to the connection having parallel torsion and curvature tensor. In other words, a T.-W parallel space is one for which the Tanaka–Webster connection $\hat{\nabla}$ is an invariant connection by parallelism.

PROPOSITION 8: *If a contact metric space satisfies $\hat{\nabla}\hat{T} = 0$, then it has an integrable associated CR structure.*

Proof. Since $\hat{\nabla}\xi = 0$, $\hat{\nabla}g = 0$ and $\hat{\nabla}\eta = 0$, it follows from (17) that

$$g((\hat{\nabla}_Z T)(X,Y),\xi) = 2g(X,(\hat{\nabla}_Z\varphi)Y) = 2g(X,\Omega(Z,Y)) = 2g(X,(\nabla_Z\varphi)Y)$$

for vector fields X and Y orthogonal to ξ. So, if $\hat{\nabla}T = 0$, then it holds $g((\nabla_Z\varphi)X, Y) = 0$ for vector fields X and Y orthogonal to ξ. Using general properties of the contact metric structure, this is equivalent to

$$(\nabla_Z\varphi)Y = g(Z + hZ, Y)\xi - \eta(Y)(Z + hZ)$$

for arbitrary vector fields Y and Z on M and hence to the integrability of the associated CR structure. ∎

Corollary 9: *A T.-W. parallel space is a homogeneous contact strongly pseudo-convex CR manifold and is analytic with respect to normal coordinate systems.*

Proof. The first claim follows at once form Kiričenko's generalization [21] of the Ambrose–Singer theorem [1], [32]. In fact, the tensor A (see (16)) gives a homogeneous structure with $\hat{\nabla}g = 0$, $\hat{\nabla}\hat{T} = 0$, $\hat{\nabla}\hat{R} = 0$, $\hat{\nabla}\xi = 0$, $\hat{\nabla}\eta = 0$ and $\hat{\nabla}\varphi = 0$.

The second claim is a consequence of [22, Theorem 7.7]. ∎

We want to find a complete classification of all T.-W. parallel spaces. First, we look at the condition $\hat{\nabla}\hat{T} = 0$.

Proposition 10: *A contact metric space satisfies $\hat{\nabla}\hat{T} = 0$ if and only if it is either Sasakian or a non-Sasakian $(k, 2)$-space.*

Proof. In general, starting from (17), we have

$$(\hat{\nabla}_Z\hat{T})(X, Y) = 2g(X, \Omega(Z, Y))\xi + \eta(Y)\Omega(Z, hX) - \eta(X)\Omega(Z, hY) + \eta(Y)\varphi(\hat{\nabla}_Z h)X - \eta(X)\varphi(\hat{\nabla}_Z h)Y. \tag{21}$$

Suppose now that $\hat{\nabla}\hat{T} = 0$. Then $\Omega = 0$ by the previous proposition and the expression (21) reduces to

$$0 = (\hat{\nabla}_Z\hat{T})(X, Y) = \eta(Y)\varphi(\hat{\nabla}_Z h)X - \eta(X)\varphi(\hat{\nabla}_Z h)Y. \tag{22}$$

Now, taking $Y = \xi$ and $X, Z \perp \xi$ and taking the inner product with $U \perp \xi$, it follows at once from (19) that

$$0 = g((\hat{\nabla}_Z h)X, U) = g((\nabla_Z h)X, U).$$

This means precisely that the contact metric space is η-parallel. So, Theorem 2 tells us that the contact metric space is either K-contact or a (non-Sasakian) (κ, μ)-space.

If it is K-contact, the integrability of the associated CR structure implies that it is actually Sasakian. If it is a non-Sasakian (k,μ)-space, then we put $Y = Z = \xi$ and $X \perp \xi$ in (22) and we use (19), (14) and (3) to obtain

$$0 = \varphi(\hat{\nabla}_\xi h)X = \varphi(\nabla_\xi h)X - 2hX = (\mu - 2)hX.$$

Clearly, this implies that $\mu = 2$.

Conversely, if M is Sasakian, then $h = \Omega = 0$ and we easily get $\hat{\nabla}\hat{T} = 0$ from (21). For a (k,μ)-space, again we have $\Omega = 0$. Further, using (19) and (14) and the fact that $h^2 = (1-k)I$ for a (k,μ)-space, we obtain $\hat{\nabla}h = 0$. Again, $\hat{\nabla}\hat{T} = 0$ follows from (21). ∎

Next, we turn our attention to the condition $\hat{\nabla}\hat{R} = 0$. First we prove the following

Proposition 11: *Let M be a (k,μ)-space. Then M satisfies $\hat{\nabla}\hat{R} = 0$ if and only if*

- *M is a locally φ-symmetric Sasakian space, or*
- *M is a non-Sasakian (k,μ)-space which is three-dimensional or for which $\mu = 2$.*

Proof. First, suppose that M is Sasakian. The Tanaka–Webster curvature is given as $\hat{R}(X,Y)Z = R(X,Y)Z + B(X,Y)Z$ (see (20)) with now

$$\begin{aligned} B(X,Y)Z = &- \eta(Z)[\eta(Y)X - \eta(X)Y] + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &+ g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z \end{aligned}$$

since $h = 0$. As the associated CR structure of a Sasakian space is integrable, we see at once that $\hat{\nabla}B = 0$. Thus, M satisfies $\hat{\nabla}\hat{R} = 0$ if and only if $\hat{\nabla}R = 0$. From (15), we get

$$\begin{aligned} (\hat{\nabla}_U R)(X,Y)Z = &(\nabla_U R)(X,Y)Z + A(U,R(X,Y)Z) - R(A(U,X),Y)Z \\ &- R(X,A(U,Y))Z - R(X,Y)A(U,Z). \end{aligned} \tag{23}$$

From (16) and (7), it follows that $A(U,X) \sim \xi$ and $g(A(U,R(X,Y)Z),V) = 0$ for any vector fields $U,X,Y,Z,V \perp \xi$. So, (23) implies that

$$g((\hat{\nabla}_U R)(X,Y)Z,V) = 0$$

for any vector fields $U,X,Y,Z,V \perp \xi$, i.e., M is a locally φ-symmetric Sasakian space.

Conversely, suppose M is Sasakian and locally φ-symmetric. In order to show that $\hat{\nabla} R = 0$, we only need to verify that $(\hat{\nabla}_U R)(X,Y)\xi = 0$ and $(\hat{\nabla}_\xi R)(X,Y)Z = 0$ for any vector fields U, X, Y, Z on M. The first one is immediate using (7). For the second one, we need an additional computation, where we use the following well-known curvature identities, valid for a Sasakian space:

$$\begin{aligned}
&R(X,Y)\varphi Z - \varphi R(X,Y)Z \\
&\qquad = g(X,Z)\varphi Y - g(Y,Z)\varphi X + g(\varphi X, Z)Y - g(\varphi Y, Z)X, \\
&R(\varphi X, Y)Z + R(X, \varphi Y)Z \\
&\qquad = g(Y,Z)\varphi X - g(X,Z)\varphi Y + g(\varphi Y, Z)X - g(\varphi X, Z)Y
\end{aligned}$$

for all vector fields X, Y, Z on M. So, from (23) we compute

$$\begin{aligned}
(\hat{\nabla}_\xi R)(X,Y)Z =& (\nabla_\xi R)(X,Y)Z + A(\xi, R(X,Y)Z) - R(A(\xi,X),Y)Z \\
& - R(X, A(\xi,Y))Z - R(X,Y)A(\xi,Z) \\
=& (\nabla_\xi R)(X,Y)Z + \varphi R(X,Y)Z - R(\varphi X, Y)Z \\
& - R(X,\varphi Y)Z - R(X,Y)\varphi Z \qquad \text{by (16)} \\
=& (\nabla_\xi R)(X,Y)Z \\
=& -(\nabla_X R)(Y,\xi)Z - (\nabla_Y R)(\xi, X)Z \\
=& R(\varphi X, Y)Z + g(\varphi X, Z)Y - g(Y,Z)\varphi X \\
& + R(X, \varphi Y)Z - g(\varphi Y, Z)X + g(X,Z)\varphi Y \qquad \text{by (7)} \\
=& 0.
\end{aligned}$$

Next, we suppose that M is a non-Sasakian (k,μ)-space. Then M satisfies $\hat{\nabla}\hat{R} = 0$ if and only if M satisfies

$$\begin{aligned}
(\hat{\nabla}_U R)(X,Y)Z =& -(\hat{\nabla}_U B)(X,Y)Z \\
=& -\eta(Z)\varphi(\hat{\nabla}_U P)(X,Y) + g(\varphi(\hat{\nabla}_U P)(X,Y), Z)\xi \\
& + \eta(Z)[\eta(Y)(\hat{\nabla}_U h)X - \eta(X)(\hat{\nabla}_U h)Y] \\
& - \eta(Y)g((\hat{\nabla}_U h)X, Z)\xi + \eta(X)g((\hat{\nabla}_U h)Y, Z)\xi \\
& - g(\varphi(\hat{\nabla}_U h)Y, Z)(\varphi X + \varphi hX) - g(\varphi Y + \varphi hY, Z)\varphi(\hat{\nabla}_U h)X \\
& + g(\varphi(\hat{\nabla}_U h)X, Z)(\varphi Y + \varphi hY) + g(\varphi X + \varphi hX, Z)\varphi(\hat{\nabla}_U h)Y
\end{aligned} \tag{24}$$

where we have, because of (14),

$$P(X,Y)$$
$$=(\nabla_X h)Y-(\nabla_Y h)X$$
$$=(1-k)\big(2g(X,\varphi Y)\xi+\eta(X)\varphi Y-\eta(Y)\varphi X\big)+(1-\mu)\big(\eta(X)\varphi hY-\eta(Y)\varphi hX\big).$$

A straightforward computation, using the explicit expression (12) for the curvature tensor, the formula (14) and the (weak) local φ-symmetry of (k,μ)-space (Theorem 3), yields that the equation (24) holds trivially, except for $U=\xi$ and $X,Y,Z\perp\xi$. For this specific case, we have

$$(\hat{\nabla}_\xi h)X=(\mu-2)h\varphi X,$$
$$(\hat{\nabla}_\xi P)(X,Y)=(1-\mu)[\eta(X)\varphi(\hat{\nabla}_\xi h)Y-\eta(Y)\varphi(\hat{\nabla}_\xi h)X]$$

and the equation (24) reduces to

$$(\hat{\nabla}_\xi R)(X,Y)Z=(\mu-2)[-g(hY,Z)(\varphi X+\varphi hX)+g(hX,Z)(\varphi Y+\varphi hY) \\ -g(\varphi Y+\varphi hY,Z)hX+g(\varphi X+\varphi hX,Z)hY].$$

Next, we use the curvature expression (12) to rewrite the left-hand side and obtain the condition

$$(25)\quad (\mu-2)[g(\varphi X,Z)hY-g(\varphi Y,Z)hX+g(hX,Z)\varphi Y-g(hY,Z)\varphi X] \\ =(\mu-2)[g(Y,Z)h\varphi X-g(X,Z)h\varphi Y-g(h\varphi X,Z)Y+g(h\varphi Y,Z)X]$$

for any vector fields $X,Y,Z\perp\xi$. Let $\{e_I=(e_i,\varphi e_i),\xi\}$, $i=1,\dots,n$ be an adapted local orthonormal frame such that $he_i=\lambda e_i$, $h\varphi e_i=-\lambda\varphi e_i$ with $\lambda=\sqrt{1-k}$ (see [6]). Then putting $X=Z=e_I$ in (25), and summing with respect to I, we obtain

$$(\mu-2)(n-1)h\varphi Y=0,$$

which yields $\mu=2$ or $\dim M=3$.

Conversely, it is easy to check that a non-Sasakian $(k,2)$-space and a three-dimensional (k,μ)-space always satisfy the condition (25). This proves the proposition. ∎

Combining Propositions 10 and 11, we now have

THEOREM 12: *A contact metric space M is a Tanaka–Webster parallel space if and only if M is a Sasakian locally φ-symmetric space or a non-Sasakian $(k,2)$-space.*

Now, the unit tangent sphere bundle of a Riemannian manifold is a (k,μ)-space if and only if the base manifold has constant curvature c [6] and then $k = c(2-c)$ and $\mu = -2c$. Also, we know that the unit tangent sphere bundle of a space of constant curvature is locally φ-symmetric [14]. So, we have

Corollary 13: *The unit tangent sphere bundle of a Riemannian manifold (M,g) is a Tanaka–Webster parallel space if and only if M has constant curvature $+1$ or -1.*

Remark 3: It was proved in [4] that the base manifold is of constant curvature $c = -1$ or $c = 1$ if and only if the standard contact Riemannian structure on the unit tangent sphere bundle is a critical point of the functional $L(g) = \int_{T_1M} Ric(\xi)\, dV$ on the set of associated Riemannian metrics $\mathcal{M}(\eta)$ of a given contact form η, where $Ric(\xi)$ denotes the Ricci curvature in the characteristic direction ξ.

Here, we define strongly locally pseudo-Hermitian symmetric spaces. Namely,

Definition 2: Let $(M;\eta,L)$ be a contact strictly pseudo-convex almost CR manifold. Then M is said to be a **strongly locally pseudo-Hermitian symmetric space** if all characteristic $\hat{\nabla}$-reflections are affine mappings, i.e., they preserve the Tanaka–Webster connection $\hat{\nabla}$.

Concerning the above property, it was proved in [3], [19] that M is strongly locally pseudo-Hermitian symmetric if and only if M satisfies the following two conditions:

$$\hat{\nabla}_X \hat{R} = \hat{\nabla}_X \hat{T} = 0$$

for any vector field $X \perp \xi$. So, among the proof of Propositions 10 and 11, we can find

Theorem 14: *A contact strictly pseudo-convex almost CR manifold M is locally pseudo-Hermitian symmetric in the strong sense if and only if M is a Sasakian locally φ-symmetric space or a non-Sasakian (k,μ)-space.*

5. Locally pseudo-Hermitian symmetric spaces

In this section, we want to define an analogue for weakly locally φ-symmetric spaces in pseudo-Hermitian geometry, i.e., with respect to the Tanaka–Webster connection. A first idea is to define them as contact metric spaces satisfying

$L((\hat{\nabla}_X \hat{R})(Y, Z)U, V) = 0$ and $L((\hat{\nabla}_X \hat{T})(Y, Z), V) = 0$ for all vector fields X, Y, Z, U, V orthogonal to ξ. However, this second condition is always satisfied, as follows at once from (21). On the other hand, it is not clear whether integrability of the CR structure follows from the above condition on the Tanaka–Webster curvature tensor alone. Since we would like to have integrability, we arrive at the following definition.

Definition 3: Let $(M; \eta, L)$ be a contact strictly pseudo-convex CR manifold. Then M is said to be a **weakly locally pseudo-Hermitian symmetric space** if M satisfies

$$L((\hat{\nabla}_X \hat{R})(Y, Z)U, V) = 0$$

for all X, Y, Z, U, V orthogonal to ξ.

From Theorem 14, we immediately have the first examples.

PROPOSITION 15: *A Sasakian manifold is a locally pseudo-Hermitian symmetric space if and only if it is locally φ-symmetric.*

PROPOSITION 16: *A non-Sasakian (k, μ)-space is a weakly locally pseudo-Hermitian symmetric space.*

In the rest of this section, we look for three-dimensional weakly locally pseudo-Hermitian symmetric spaces. For that purpose, we adopt the notation of [15] for three-dimensional contact strictly pseudo-convex CR manifolds. So, we consider on M the maximal open set U_1 on which $h \neq 0$ and the maximal open subset U_2 on which h is identically zero. Suppose that M is non-Sasakian. Then U_1 is non-empty and there is a local orthonormal frame field $\{\xi, e, \varphi e\}$ on U_1 such that $h(e) = \lambda e$, $h(\varphi e) = -\lambda \varphi e$ for some positive function λ. The covariant derivative is then of the following form (see [15, Lemma 2.1]):

$$\begin{aligned}
&\nabla_\xi \xi = 0, \qquad \nabla_\xi e = -a\varphi e, \qquad \nabla_\xi \varphi e = ae,\\
&\nabla_e \xi = -(\lambda+1)\varphi e, \qquad \nabla_{\varphi e}\xi = (1-\lambda)e,\\
&\nabla_e e = \frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}\varphi e, \qquad \nabla_{\varphi e}\varphi e = \frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}e,\\
&\nabla_e \varphi e = -\frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}e + (\lambda+1)\xi,\\
&\nabla_{\varphi e} e = -\frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}\varphi e + (\lambda-1)\xi.
\end{aligned} \tag{26}$$

Here, a is a smooth function and $\sigma = \rho(\xi, \cdot)$ where ρ denotes the Ricci tensor. By using (16) we also calculate the (1,2)-tensor field A:

$$\begin{aligned} &A(\xi,\xi) = 0, && A(e,\xi) = (1+\lambda)\varphi e, && A(\varphi e,\xi) = -(1-\lambda)e, \\ &A(\xi,e) = \varphi e, && A(e,e) = 0, && A(\varphi e,e) = (1-\lambda)\xi, \\ &A(\xi,\varphi e) = -e, && A(e,\varphi e) = -(1+\lambda)\xi, && A(\varphi e,\varphi e) = 0. \end{aligned} \tag{27}$$

Using (26) and (27), we get the following expressions for $\hat{\nabla}$ from (15):

(28)

$$\begin{aligned} &\hat{\nabla}_\xi \xi = 0, && \hat{\nabla}_\xi e = (1-a)\varphi e, && \hat{\nabla}_\xi \varphi e = -(1-a)e, \\ &\hat{\nabla}_e \xi = 0, && \hat{\nabla}_e e = \frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}\varphi e, && \hat{\nabla}_e \varphi e = -\frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}e, \\ &\hat{\nabla}_{\varphi e} \xi = 0, && \hat{\nabla}_{\varphi e} e = -\frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}\varphi e, && \hat{\nabla}_{\varphi e} \varphi e = \frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}e. \end{aligned}$$

From this, we calculate the Tanaka–Webster curvature tensor:

$$\begin{aligned} \hat{R}(\xi,e)e =& \hat{\nabla}_\xi \hat{\nabla}_e e - \hat{\nabla}_e \hat{\nabla}_\xi e - \hat{\nabla}_{[\xi,e]} e \\ =& \big(\xi(A) + e(a) + (1+\lambda-a)B\big)\varphi e, \\ \hat{R}(\xi,e)\varphi e =& -\big(\xi(A) + e(a) + (1+\lambda-a)B\big)e, \\ \hat{R}(\xi,\varphi e)e =& -\big(\xi(B) - \varphi e(a) - (1-\lambda-a)A\big)\varphi e, \\ \hat{R}(\xi,\varphi e)\varphi e =& \big(\xi(B) - \varphi e(a) - (1-\lambda-a)A\big)\varphi e, \\ \hat{R}(e,\varphi e)e =& -\big(e(B) + \varphi e(A) + 2(1-a)\big)\varphi e, \\ \hat{R}(e,\varphi e)\varphi e =& \big(e(B) + \varphi e(A) + 2(1-a)\big)e, \\ \hat{R}(\cdot,\cdot)\xi =& 0. \end{aligned} \tag{29}$$

where we have put $A = \frac{1}{2\lambda}(\varphi e(\lambda) + \sigma(e))$, $B = \frac{1}{2\lambda}(e(\lambda) + \sigma(\varphi e))$. From (28) and (29) we obtain

$$\begin{aligned} (\hat{\nabla}_e \hat{R})(e,\varphi e)e =& \hat{\nabla}(\hat{R}(e,\varphi e)e) \\ & - \hat{R}(\hat{\nabla}_e e,\varphi e)e - \hat{R}(e,\hat{\nabla}_e \varphi e)e - \hat{R}(e,\varphi e)\hat{\nabla}_e e \\ =& -e\big(e(B) + \varphi e(A) + 2(1-a)\big)\varphi e, \end{aligned} \tag{30}$$

$$(\hat{\nabla}_{\varphi e} \hat{R})(e,\varphi e)e = -\varphi e\big(e(B) + \varphi e(A) + 2(1-a)\big)\varphi e. \tag{31}$$

So, if M is weakly locally pseudo-Hermitian symmetric, then it holds

$$\begin{aligned} 0 =& e\big(e(B) + \varphi e(A) + 2(1-a)\big), \\ 0 =& \varphi e\big(e(B) + \varphi e(A) + 2(1-a)\big). \end{aligned}$$

In that case, we also have, using the second condition of (2):

$$\begin{aligned} 0 =& [e, \varphi e]\big(e(B) + \varphi e(A) + 2(1-a)\big) \\ =& \eta([e, \varphi e])\xi\big(e(B) + \varphi e(A) + 2(1-a)\big) \\ =& - d\eta(e, \varphi e)\xi\big(e(B) + \varphi e(A) + 2(1-a)\big) \\ =& 2\xi\big(e(B) + \varphi e(A) + 2(1-a)\big). \end{aligned}$$

Thus, we have

Proposition 17: *A three-dimensional contact metric space $(M; \eta, g)$ is locally pseudo-Hermitian symmetric if and only if it is locally φ-symmetric on U_2 and $e(B) + \varphi e(A) + 2(1-a)$ is constant on U_1.*

In particular, any three-dimensional contact-homogeneous contact metric space is weakly locally pseudo-Hermitian, since A, B and a are constant in that case (see [25]). Using this proposition, we now illustrate that the class of weakly locally φ-symmetric spaces and the class of weakly locally pseudo-Hermitian symmetric spaces are essentially different.

Example 1: Non-unimodular Lie groups G with left invariant contact metric structures: From [23] and [25], we know that there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{g}$ such that

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = \gamma e_2,$$

where α, γ are constants and $\alpha \neq 0$. We have

$$he_1 = 1/2(L_\xi \varphi)e_1 = (\gamma/2)e_1, \quad he_2 = -(\gamma/2), e_2.$$

This corresponds to $a = (2-\gamma)/2$, $\lambda = \gamma/2$, $A = 0$, $B = \alpha$. From this, we get that $e(B) + \varphi e(A) + 2(1-a) = \gamma$, constant. Applying Proposition 17, we see that G is locally pseudo-Hermitian symmetric for all values of α and γ. On the other hand, it was shown in [11] that the non-unimodular Lie group G is weakly locally φ-symmetric if and only if $\gamma = 0$ (the Sasakian case) or $\gamma = -2$.

Example 2: Perrone's non-homogeneous three-dimensional weakly locally φ-symmetric example M_1 ([26]): Let $M_1 = \{(x, y, z) \in \mathbf{R}^3(x, y, z) : \ x \neq 0\}$ be the contact three-manifold endowed with the contact form $\eta = xydx + dz$. Its characteristic vector field is given by $\xi = \partial/\partial z$. Take a global frame field

$$e_1 = -\frac{2}{x}\frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}, \quad e_3 = \xi$$

and define a Riemann metric g such that $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. Moreover, we define φ by $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi\xi = 0$. Then (η, φ, ξ, g) is a contact metric structure. The structure operator h satisfies $he_1 = e_1$, $he_2 = -e_2$. In this case, $a = 2$, $\lambda = 1$, $A = -1/x$, $B = 0$. This yields $e(B) + \varphi e(A) + 2(1-a) = 1/x^2 - 2$, which is not constant. Hence, the space M_1 is weakly locally φ-symmetric, but is not locally pseudo-Hermitian symmetric.

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